ENTIRE VECTORS AND HOLOMORPHIC EXTENSION OF REPRESENTATIONS

BY

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ABSTRACT. Let G be a connected, simply connected real Lie group and let U be a representation of G in a complete, locally convex, topological vector space $\mathcal{T}$. If G is solvable, it can be canonically embedded in its complexification G_c. A vector $v \in \mathcal{T}$ is said to be entire for U if the map $g \to U_g v$ of G into $\mathcal{T}$ is holomorphically extendible to G_c. The space of entire vectors is an invariant subspace of the space of analytic vectors. U is said to be holomorphically extendible iff the space of entire vectors is dense.

In this paper we consider the question of existence of holomorphic extensions We prove

Theorem. *A unitary representation U is holomorphically extendible to G_c iff G modulo the kernel of U is type R in the sense of Auslander-Moore* [1].

In the process of proving the above results, we develop several interesting characterizations of entire vectors which generalize work of Goodman for solvable Lie groups and we prove a conjecture of Nelson concerning the relationship between infinitesimal representations of Lie algebras and representations of the corresponding Lie groups.

Introduction. Let G be a connected, simply connected (c.s.c.) real solvable Lie group with Lie algebra $\mathcal{L}$. Let $\mathcal{L}_c$ denote the complexification of $\mathcal{L}$ and let G_c be the corresponding c.s.c. complex Lie group. From the solvability of G, G is canonically imbeddable as a closed subgroup of G_c. If $f: G \to \mathcal{T}$ is a mapping of G into a complete locally convex topological vector space $\mathcal{T}$, we will say that f is holomorphically extendible to G_c if there is a continuous mapping $f_c: G_c \to \mathcal{T}$ which extends f such that the map $z \to \langle f_c(z), w \rangle$ is a holomorphic map of G_c into C for all $w \in \mathcal{T}'$ (the space of continuous linear maps of $\mathcal{T}$ into C). It can be shown that in this case the map f_c is a differentiable map of G_c into $\mathcal{T}$ in the sense that $\lim_{t \to 0}(f(z \exp tX) - f(z))/t$ exists for all $X \in \mathcal{L}_c$ and $z \in G_c$ and defines a continuous function of z. (See §III.8 of Grothendieck [5] for details.)

If U is a strongly continuous representation of G in $\mathcal{T}$ by continuous operators, we define, following Goodman [2], $\mathcal{H}_\infty^\omega(U)$ (which we also denote by $\mathcal{H}_\infty^\omega$ when sufficiently clear) to be the space of vectors $v \in \mathcal{T}$ for which the map $g \to U_g v$ of G into $\mathcal{T}$ is holomorphically extendible to G_c. $\mathcal{H}_\infty^\omega$ is an invariant subspace of

$\mathcal{T}$ and the restriction of U to $\mathcal{H}_\infty^\omega$ is extendible to an analytic representation of G_c in $\mathcal{H}_\infty^\omega$. When $\mathcal{H}_\infty^\omega$ is dense in $\mathcal{T}$ we shall say that U has an analytic extension and we will call the extension of the part of U in $\mathcal{H}_\infty^\omega$ the analytic extension of U. We shall denote it by U^c or simply U when clear.

In this paper we are interested in obtaining conditions under which the analytic extensions exist. The structure of the paper is as follows:

In §I we develop several infinitesimal characterizations of entire vectors which generalize results of Goodman [3] in the nilpotent-unitary case and of Nelson [8] in the analytic vector case. We also remark that there is a simple proof of a conjecture of Nelson concerning the relationship of infinitesimal representations of Lie algebras and representations of the corresponding Lie groups.

In §II, we give a complete characterization of those unitary representations which have analytic extensions. We also obtain sufficient conditions for a general representation to have entire vectors. These conditions imply, in particular, that every Banach space representation of a type R solvable Lie group has a dense set of entire vectors. Our methods are analogous to those used by Robert Moore for analytic vectors [7].

0. **Notation.** Throughout the rest of this paper we shall assume that the notation established above (e.g., G, $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{L}$, etc.) remains in force. In addition, if K is a Lie group, $\mathcal{L}(K)$ will denote its Lie algebra.

N will denote the natural numbers.

If $a_1, \ldots, a_p$ are elements of some associative algebra and $(n_1, \ldots, n_p) = n \in N^p$, then $(a_1, \ldots, a_p)^n = a_1^{n_1} \cdots a_p^{n_p}$. We adopt the usual conventions concerning multi-indices—e.g. $n! = n_1! \cdots n_p!$, $|n| = \sum n_i$, etc.

We define $C^n(U)$ to be the space of vectors v in $\mathcal{T}$ such that $g \to U(g)v$ is a C^n map of G into $\mathcal{T}$ and $C^\infty(U) = \cap_{n=1}^\infty C^n(U)$. For $v \in C^1(U)$, and $X \in \mathcal{L}$, define $\partial U(X)v = \lim_{t\to 0}(U(\exp tX)v - v)/t$.

If $X = (X_1, \ldots, X_n) \in \mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L}$, we set $\partial U(X) = (\partial U(X_1), \ldots, \partial U(X_n))$.

I. Infinitesimal characterizations. Let $X = (X_1, \ldots, X_d)$ be an ordered Jordan Hölder basis for $\mathcal{L}$. (Recall that an ordered basis $(X_1, \ldots, X_d)$ is an ordered Jordan Hölder basis for $\mathcal{L}$ if $[X_i, X_{i+1}] \equiv 0 \bmod \{X_{i+1}, \cdots, X_n\}$. Every solvable Lie algebra has such a basis.)

We have the following characterization of entire vectors.

Proposition. (I.1). $v \in C^\infty(U)$ is an entire vector for U iff the family of sets $S_{v,t} = \{(U(X)/n!)t^{|n|}v | n \in N^d\}$ is bounded for all $t > 0$.

Proof. Since $\{X_1, iX_1, X_2, iX_2, \ldots, X_d, iX_d\}$ is a Jordan Hölder basis for $\mathcal{L}_c$, the map $(z_1, \ldots, z_d) \to \exp z_1 X_1 \cdots \exp z_d X_d$ of C^d into G_c provides global holomorphic coordinates for G_c.

If v is entire for U and if $w \in \mathcal{T}'$, it follows from the Cauchy estimates for $(z_1, \ldots, z_d) \to \langle U(\exp z_1 X_1 \cdots \exp z_d X_d)v, w\rangle$ that $\langle S_{v,t}, w\rangle$ is bounded for all $t > 0$. Thus, $S_{v,t}$ is weakly, and hence strongly, bounded.

Conversely, suppose $S_{v,t}$ is bounded for all $t > 0$. Consider, for $z \in C^d$, the series

$$\phi(z) = \sum_{n \in N^d} \frac{z^n}{n!} \partial U(X)^n v.$$

(Recall $X = (X_1, \ldots, X_d)$.) By standard power series estimates, $\phi(z)$ is Cauchy in any continuous seminorm on $\mathcal{T}$ and hence converges in $\mathcal{T}$ and defines a holomorphic map of C^d into $\mathcal{T}$. We claim that for $t = (t_1, \ldots, t_d) \in R^d$,

$$\phi(t_1, \ldots, t_d) = U(\exp t_1 X_1) U(\exp t_2 X_2) \cdots U(\exp t_d X_d)v.$$

If $t_2 = t_3 = \cdots = t_d = 0$, this follows from Taylor's theorem, for let $w \in \mathcal{T}'$. Then

(1)
$$\langle U(\exp t_1 X_1)v, w \rangle - \sum_0^K \left\langle \frac{\partial U(X_1)}{n!} v t_1^n \right\rangle$$
$$= \int_0^{t_1} \langle U(\exp S X_1) \partial U(X_1)^{K+1} v, w \rangle \frac{(t_1 - S)^K}{K!} dS.$$

The integrand is bounded independently of K. In fact, if $u \in \mathcal{T}'$, the family of linear functionals $u \circ U(\exp S X_1)$ as S ranges from 0 to t_1, is pointwise $(\sigma(E', E))$ bounded and hence strongly $(\beta(E', E))$ bounded. Thus if B is any bounded set, $\bigcup_{0 \le S \le t} U(\exp SX)B$ is weakly, and hence strongly, bounded. Applying this reasoning with $B = \{ \partial U(X_1)^n / n! \, (2t_1)^n v \mid n \in N \}$, we see

$$\left| \int_0^{t_1} \langle U(\exp SX) \partial U(X_1)^{K+1} v, w \rangle \frac{(t_1 - S)^K}{K!} dS \right|$$
$$\le M \frac{K+1}{(2t_1)^{K+1}} \int_0^{t_1} (t_1 - S)^K dS = M2^{-(K+1)}.$$

Thus (1) above converges to zero and $\phi(t_1, 0, \ldots, 0) = U(\exp t_1 X_1)v$. The same reasoning applies to the vector $\partial U(X_2)^{n_2} \cdots \partial U(X_d)^{n_d} v$ (n_i fixed). Thus if $\overline{X} = (X_2, \ldots, X_d)$ and $t = (t_1, \ldots, t_d) \in R^d$, then

$$\langle \phi(t), w \rangle = \sum_{n \in N^d} \frac{t^n}{n!} \langle \partial U(X)^n v, w \rangle$$
$$= \sum_{\overline{n} \in N^{d-1}} \frac{t^{\overline{n}}}{\overline{n}!} \sum_{n=0}^{\infty} \frac{t_1^n}{n!} \langle \partial U(X_1)^n \partial U(\overline{X})^{\overline{n}} v, w \rangle$$
$$= \sum_{\overline{n} \in N^{d-1}} \frac{t^{\overline{n}}}{(\overline{n})!} \langle U(\exp t_1 X_1) \partial U(\overline{X})^{\overline{n}} v, w \rangle$$
$$= \sum_{\overline{n} \in N^{d-1}} \frac{t^{\overline{n}}}{(\overline{n})!} \langle \partial U(\overline{X})^{\overline{n}} v, U'(\exp t X_1)(w) \rangle$$

where $'$ denotes adjoint. Continuing by induction we see that

$$\langle \phi(t), w \rangle = \langle U(\exp t_1 X) U(\exp t_2 X_2) \cdots U(\exp t_d X_d)v, w \rangle,$$

i.e.

$$\phi(t) = U(\exp t_1 X_1 \cdots \exp t_d X_d)v. \quad \text{Q.E.D.}$$

Remark. In the above proof we used only that the "matrix elements" $g \to \langle U(g)v, w \rangle$ were holomorphically extendible to G_c. Also, we concluded that $g \to U(g)v$ was expandable, in terms of our global coordinates, in a convergent power series about 0. Hence we have

Corollary (I.2). *The following are equivalent:*
(a) *v is entire;*
(b) *$g \to \langle U_g v, w \rangle$ is holomorphically extendible to G_c for all $w \in \mathcal{T}'$;*
(c) *$g \to U_g v$ is expandable in terms of global coordinates in an absolutely convergent power series.*

Entire vectors satisfy a seemingly stronger property than that of (I.1).

Theorem (I.3). *Let $\mathcal{R}$ be a finite subset of $\mathcal{L}$. If $v \in \mathcal{H}_\infty^\omega$, then for all $t > 0$ the set*

$$\mathcal{R}_{v,t} = \left\{ \frac{A_1 \cdots A_K}{K!} vt^K \mid A_i = \partial U(R_i), R_i \in \mathcal{R}, K \in N \right\}$$

is bounded in $\mathcal{T}$.

The proof is a counting argument utilizing the commutators of the A_i. The main algebraic step is contained in the following lemma.

Let $\mathcal{A}$ be an associative algebra. If $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $\mathcal{A}$, let $\mathcal{X}^n$ denote the set of products of n elements from $\mathcal{X}$ and let $(\mathcal{X}, \mathcal{Y})^{n,m}$ denote the set of products of $n + m$ elements from $\mathcal{X} \cup \mathcal{Y}$, n of which come from $\mathcal{X}$ and m of which come from $\mathcal{Y}$. For $y \in \mathcal{A}$, let ad $y: \mathcal{A} \to \mathcal{A}$ be the map ad $y(x) = xy - yx$.

Lemma 1. *If $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $\mathcal{A}$, then every $a \in (\mathcal{X}, \mathcal{Y})^{n,m}$ is expressible in the form $a = \sum_{K=0}^m a_K$ where each a_K is a sum of at most $n^K \binom{m}{K}$ terms of the form*

$$(2) \qquad \begin{aligned} y_1' \cdots y_{m-K}' \cdot [\text{ad } y_1 \cdots \text{ad } y_{K_1}(x_1)][\text{ad } y_{K_1+1} \cdots \text{ad } y_{K_2}(x_2)] \\ \cdots [\text{ad } y_{K_{n-1}+1} \cdots \text{ad } y_K(x_n)] \end{aligned}$$

where y_i' and $y_i \in \mathcal{Y}$, $x_i \in \mathcal{X}$ and where $0 \le K_1 \le K_2 \le \cdots \le K_{n-1} \le K$. (If $K_j = K_{j+1}$, then we take ad $y_{K_j+1} \cdots$ ad $y_{K_{j+1}}(x_{j+1}) = x_{j+1}$. We treat the cases $K_1 = 0$ and $K_{n-1} = K$ similarly.)

Proof. Let $Z(n, m, K)$ be the set of terms of the form of (2) above union the set $\{0\}$. The lemma asserts that $(\mathcal{X}, \mathcal{Y})^{n,m} \subset \sum_{K=0}^m m^K \binom{m}{K} Z(n, m, K)$. The relations $Z(n, 0, 0) = \mathcal{X}^n$ and $Z(0, m, 0) = \mathcal{Y}^m$ imply that the lemma is true if either $m = 0$ or $n = 0$. Assume that the lemma is false and let (n, m) be the smallest pair of integers for which it is false relative to the ordering $(a, b) \le (c, d)$ iff $a \le c$ and $b \le d$. Note that $n > 0$ and $m > 0$. Let $a \in (\mathcal{X}, \mathcal{Y})^{n,m}$. Either

$a = a'x$ where $x \in \mathcal{X}$ and $a' \in (\mathcal{X}, \mathcal{Y})^{n-1,m}$ or $a = a'y$ where $a' \in (\mathcal{X}, \mathcal{Y})^{n,m-1}$ and $y \in \mathcal{Y}$. In the first case, the relation $Z(n-1, m, K) \cdot \mathcal{X} \subset Z(n, m, K)$ and the inductive hypothesis imply that the lemma is true for a. In the second case, the relation $Z(n, m-1, K) \cdot y \subset nZ(n, m, K+1) + Z(n, m, K)$ (which follows easily from the identity $v_1 v_2 \cdots v_n \cdot y = \sum_{j=1}^{n} v_1 \cdots \mathrm{ad}\, y(v_j) \cdots v_n + yv_1 \cdots v_n$) and the inductive hypothesis imply that

$$a \in \sum_{K=0}^{m-1} n^K \binom{m-1}{K} nZ(n, m, K+1) + \sum_{K=0}^{m-1} n^K \binom{m-1}{K} Z(n, m, K).$$

Regrouping and using the identity

$$\binom{m-1}{K-1} + \binom{m-1}{K} = \binom{m}{K}$$

we see that the lemma is true for a. This contradiction proves the lemma. Q.E.D.

For the next lemma, let $\mathcal{C}$ be a vector space and let μ be a Minkowski functional on $\mathcal{C}$ corresponding to a balanced but not necessarily absorbing convex set (i.e. μ satisfies $\mu(x+y) \leq \mu(x) + \mu(y)$ and $\mu(\alpha y) = |\alpha|\mu(y)$ but μ may assume infinite values). Let $\mathcal{A}$ be an algebra of linear operators on $\mathcal{C}$ and let $\mathcal{N}$ be a finite dimensional vector subspace of $\mathcal{A}$ with ordered basis $\mathcal{X} = (x_1, \ldots, x_d)$. Let $\mathcal{Y}$ be a finite subset of $\mathcal{A}$ for which $\mathrm{ad}\, y$ leaves $\mathcal{N}$ invariant for $y \in \mathcal{Y}$. Then

Lemma 2. *If $v \in \mathcal{C}$ is such that*

(3)
$$\left\{ \frac{y}{m!} \cdot \frac{x}{n!}(v)t^{n+m} \mid x \in \mathcal{X}^n, y \in \mathcal{Y}^m, n, m \in N \right\}$$

is bounded in μ for all $t > 0$, then $\{(a/K!)(v)t^K \mid a \in (\mathcal{X} \cup \mathcal{Y})^K, K \in N\}$ is bounded in μ for all $t > 0$.

Proof. Let $a \in (\mathcal{X} \cup \mathcal{Y})^j$. Then $a \in (\mathcal{X}, \mathcal{Y})^{n,m}$ for some $n, m \in N$, $n + m = j$. Write $a = \sum_{K=0}^{m} a_K$ where the a_K are as in Lemma 1. We estimate $\mu(a_K(v))$ by estimating terms of the form of (2) above. In particular, following Goodman [2], we put on $\mathcal{N}$ the norm $|\sum_{i=1}^{d} c_i x_i| = \sum_{i=1}^{d} |c_i|$. Since $\mathcal{N}$ is finite dimensional, $\mathrm{ad}\, y$, $y \in \mathcal{Y}$, defines a bounded linear transform (with respect to $|\cdot|$) of $\mathcal{N}$ into itself. Let M be a common bound for these transforms.

Also, it is easily shown that if $n_1, \ldots, n_q \in \mathcal{N}$ and $a \in \mathcal{A}$, then

$$\mu(an_1 \cdots n_q(v)) \leq |n_1| \cdots |n_q| \sup_{x \in \mathcal{X}^q} \mu(ax(v)).$$

Thus, if b is any term of the form of (2) above, $t \geq r > 0$ and C_t is a bound on the set (3) above, then

$$\mu(b(v)) \leq M^K \sup_{x \in \mathfrak{X}^n} \mu(y'_1 \cdots y'_{m-K} x(v))$$

$$\leq M^K \sup_{y \in \mathcal{Y}^{m-K}, x \in \mathfrak{X}^n} \mu\left(\frac{y}{(m-K)!} \frac{x}{n!}(v) t^{n+(m-K)}\right) \cdot t^{-(n+(m-K))} n! \, (m-K)!$$

$$\leq M^K C_t t^{-n} r^{-(m-K)} n! \, (m-K)!.$$

Thus,

$$\mu(a(v)) \leq \sum_{K=0}^{m} \mu(a_K(v))$$

$$\leq \sum_{K=0}^{m} n^K \binom{m}{K} M^K C_t t^{-n} r^{-(m-K)} n! \, (m-K)!$$

$$\leq C_t t^{-n} r^{-m} n! \, m! \sum_{K=0}^{\infty} \frac{(nMr)^K}{K!}$$

$$= C_t (t e^{-Mr})^{-n} r^{-m} n! \, m!$$

$$\leq C_t (e^{Mr}/t + 1/r)^{n+m} 2^{n+m} j!$$

$$= C_t \left(\frac{2rt}{re^{Mr} + t}\right)^{-j} j!.$$

This implies the lemma since $2rt/(re^{Mr} + t)$ can be made arbitrarily large.

To prove (I.3) let ρ be a Minkowski functional based on a not necessarily absorbing convex set in $\mathcal{J}$. We shall show that if the $S_{v,t}$ of (I.1) are bounded in ρ for all $t > 0$, then the $\mathcal{R}_{v,t}$ are also bounded in ρ. If dim $G = 1$, this is clear, so suppose $d = \dim G > 1$ and the above assertion is true for all groups of lower dimensions and all such ρ.

For $s > 0$ and $w \in C^\infty(U)$, define

$$\mu_s(w) = \sup_{n \in N} \rho\left(\frac{\partial U(X_1)^n}{n!} w s^n\right).$$

μ_s is a Minkowski functional and, letting $\overline{X} = (X_2, \ldots, X_d)$, $\{(\partial U(\overline{X})^n/n!)vt^{|n|} \mid n \in N^{d-1}\}$ is bounded in μ_s. Since $(X_1, \ldots, X_d)$ is a Jordan Hölder basis, span$\{X_2, \ldots, X_d\}$ is an ideal in $\mathcal{L}$. It follows then from the inductive hypothesis that if $\mathfrak{X} = \{\partial U(X_2), \ldots, \partial U(X_d)\}$, then $\{\mu_s((A(v)/n!)t^n) \mid A \in \mathfrak{X}^n, n \in N\}$ is bounded—i.e.

$$\left\{\frac{\partial U(X_1)^{n_1}}{n_1!} \frac{A(v)}{n!} s^{n_1} t^n \mid n_1, n \in N, A \in \mathfrak{X}^n\right\}$$

is ρ bounded. It now follows from Lemma 2 that (I.2) is true with $\mathcal{R} = \{X_1, \ldots, X_d\}$. (I.2) for general $\mathcal{R}$ follows easily from this.   Q.E.D.

As corollaries to the above proof we have the following facts, all due to Goodman in the nilpotent case. Our proofs are inspired by Goodman's.

Corollary (I.4.) *Let U be a unitary representation of G in a Hilbert space $\mathcal{H}$. If G is a product of two analytic subgroups K and N with N normal then v is entire for U if (and only if) v is entire for $U\mid_N$ and $U\mid_K$.*

Proof. Let Y and X be, respectively, bases for $\mathcal{L}(K)$ and $\mathcal{L}(N)$. Let $\mathcal{Y} = \partial U(Y)$ and $\mathcal{X} = \partial U(X)$, considered as sets of operators on $\mathcal{C} = C^\infty(U)$. Let $\mathcal{A}$ be the algebra they generate and let μ be the seminorm $|(\cdot, v)|$. If $y \in \mathcal{Y}^n$ and $x \in \mathcal{X}^m$, $n, m \in \mathcal{N}$, then from the unitarity of U, $y^* \in \mathcal{Y}^n$ and

$$\mu\left(\frac{y}{n!}\frac{x}{m!}(v)t^{n+m}\right) = \left|\left(\frac{x}{m!}(v)t^m, \frac{y^*}{n!}(v)t^n\right)\right| \leq \left\|\frac{x}{m!}(v)t^m\right\| \left\|\frac{y^*}{n!}(v)t^n\right\|.$$

By (I.3) this is bounded independently of m and n. Thus, by Lemma 2 above, for all $t > 0$ there is a constant $C_t > 0$ such that if $a \in (\mathcal{X} \cup \mathcal{Y})^K$, then $\mu(a(v)) \leq C_t K! \, t^{-K}$. Hence, if $a \in (\mathcal{X} \cup \mathcal{Y})^K$, then

$$t^{-2K}(2K)! \, C_t \geq \mu(a^* a(v)) = \|a(v)\|^2.$$

The entirety of v follows from (I.2). Q.E.D.

Corollary (I.5). *If U is as in Corollary (I.4) then v is entire for U iff v is entire for each of the representations $t \to U(\exp tX_i)$, $i = 1, \ldots, d$, where $(X_1, \ldots, X_d)$ is an ordered Jordan Hölder basis.*

Proof. Follows by induction from Corollary (I.4).

Corollary (I.6). *If U is as in Corollary (I.4), then v is entire for U iff $g \to (U_g v, v)$ is holomorphically extendible to G_c.*

Proof. Reasoning as in the "only if" part of (I.1), it is easily seen that for $X \in \mathcal{L}$, $t > 0$, $\{(\partial U(X)^n/n!)(v)t^n \mid n \in N\}$ is bounded in the seminorm $|(\cdot, v)|$. Hence, as in the proof of Corollary (I.5), v is entire for $t \to U(\exp tX)$ from (I.1). Since X was arbitrary, the conclusion follows from Corollary (I.5). Q.E.D.

Remark (I.7). The conclusion of (I.3), since it requires neither the simple-connectedness nor solvability of G, is usually taken as the definition of entirety of v. We shall henceforth assume (I.3) as the definition of entirety. Note, however, that by lifting to the universal covering group, one may assume simple-connectedness without loss of generality.

Note, also, that (I.3) establishes that entire vectors are a type of analytic vector (see Nelson [8]). In fact, if in (I.3), $\mathcal{R} = \{X_1, \ldots, X_d\}$ is *any* basis of $\mathcal{L}$, then the analytic vectors for U are the set of v such that $\mathcal{R}_{v,t}$ is bounded for some $t > 0$. As Nelson noted, this definition depends only on the representation of $\mathcal{L}$ given by $X \to \partial U(X)$ and not on U.

In [8], Nelson showed that a representation ρ of a Lie algebra $\mathfrak{G}$ by a family of partially defined skew symmetric operators in a Hilbert space $\mathcal{H}$ having a common, dense, invariant domain is the restriction of the differential of a unitary representation of the corresponding c.s.c. Lie group G_0 iff there is a $t > 0$ such

that the space of vectors v for which $\mathcal{R}_{v,t}$ is bounded (denoted $\mathcal{H}_t$) is dense in $\mathcal{H}$. Nelson remarked that it was unknown if it is sufficient to assume only that the space of analytic vectors is dense. We remark that it is indeed sufficient.

In fact, let $X = (X_1, \ldots, X_d)$ be a basis for $\mathfrak{G}$ and let $\Omega \subset R^d$ be an open neighborhood of 0 for which the map $\phi(t_1, \ldots, t_d) \to \exp t_1 X_1 \cdots \exp t_d X_d$ is a homeomorphism of Ω onto an open subset Ω' in G_0. In [8] Nelson showed that under the assumption of density of the analytic vectors the $\rho(Y)$ are essentially skew-adjoint for $Y \in \mathfrak{G}$. Hence for $t \in R$ we may form the operator $\exp t \overline{\rho(Y)}$ which is unitary by Stone's theorem, and hence we may define a mapping U of Ω' into unitary operators via

$$\exp t_1 X_1 \cdots \exp t_d X_d \to \exp t_1 \overline{\rho(X_1)} \cdots \exp t_d \overline{\rho(X_d)}$$

for $t = (t_1, \ldots, t_d) \in \Omega$. If U can be shown to be a local representation of Ω', it will follow as in Nelson that U is extendible to a unitary representation of G_0 whose differential extends ρ. Now let v be an analytic vector for ρ, say $v \in \mathcal{H}_t$. It is easily seen that $\mathcal{C}_v = \text{span}\{\rho(Y_1) \cdots \rho(Y_K)v \mid Y_i \in \mathcal{R}, K \in N\}$ is a ρ invariant subspace of $\mathcal{H}_s$ for any $s < t$. Let $\mathcal{H}_v = \overline{\mathcal{C}_v}$ and let $\tilde{\rho}$ be the representation of $\mathfrak{G}$ defined by $\tilde{\rho}(X) = \rho(X)\mid_{\mathcal{C}_v}$ considered as a partially defined operation on $\mathcal{H}_v$. By Nelson's theorem the map

$$\exp t_1 X_1, \ldots, \exp t_d X_d \to \exp t_1 \overline{\tilde{\rho}(X_1)} \cdots \exp t_d \overline{\tilde{\rho}(X_d)}$$

for $t = (t_1, \ldots, t_d) \in \Omega$ defines a local representation U of Ω' in $\mathcal{H}_v$. Hence, to prove that U is a local representation, it suffices to show that the restriction of U to $\mathcal{H}_v$ is equal to $\tilde{U}$ for all $v \in \mathcal{H}_t$. But, if $w \in \mathcal{C}_v$ it can be shown via the reasoning of Theorem (I.1) that $\tilde{U}(\phi(t))w$ and $U(\phi(t))w$ are both given by the same power series for t close enough to zero (independently of w). Since $\mathcal{C}_v$ is dense in $\mathcal{H}_v$, our remark is proven.

It should be noted that it now follows from the converse of Nelson's theorem that $\mathcal{H}_t$ is in fact dense for some $t > 0$.

Example (I.8). Let G^A be the two dimensional Lie group with Lie algebra spanned by X and Y satisfying $[X, Y] = \alpha_0 Y$. It is well known that every infinite dimensional irreducible unitary representation of G^A is realizable in $L^2(R)$ in such a way that $\partial U(X) = \partial/\partial t$ and $\partial U(Y) = i\lambda e^{\alpha_0 t}, \lambda$ and α_0 some real constant. We remark for the purposes of Example (I.9) that for every λ and $\alpha_0 \neq 0$ in R there is such a representation. Goodman showed that no such representation could have nonzero entire vectors. From this, he was able to show, among other things, that no locally injective representation of G^A could have entire vectors. See [2] for this.

Example (I.9). Let $\alpha = \alpha_0 + \alpha_1 i \in C$ and let G^α be the group structure on $C \times R$ defined by $(c, r) * (c', r') = (c + e^{\alpha r} c', r + r')$. G^α is a solvable, connected, simply connected Lie group. Let λ be a nonzero complex number. Let U^λ be the representation of G^α in $L^2(R)$ given by

$$U^\lambda_{(c,r)}f(t) = \exp i \, \mathrm{Re}(\lambda e^{\alpha t}c)f(t + r).$$

It can be shown via Mackey's theory of induced representations that for $\mathrm{Re}\,\alpha \neq 0$, U^λ is irreducible and every irreducible representation of G^α is of this form. The Lie algebra of G^α is isomorphic to $C \times_S R$ with the commutation rule $[(c,r),(c',r')] = [\alpha(cr' - rc', 0)]$. Let $Y_1 = (i,0)$, $Y_2 = (1,0)$ and $X = (0,1)$. Then

$$\partial U(Y_1)f(t) = -i \, \mathrm{Im}(\lambda e^{\alpha t})f(t), \qquad \partial U(Y_2) = i \, \mathrm{Re}(\lambda e^{\alpha t})f(t)$$

and

$$\partial U(X_1)f(t) = (\partial/\partial t)f(t).$$

We claim U^λ has no nonzero entire vectors. Suppose f is entire for U^λ. Let

$$Y = \sqrt{\partial U(Y_1)^2 + \partial U(Y_2)^2} = i|\lambda e^{\alpha t}| = i|\lambda|e^{\alpha_0 t}.$$

From the inequalities $(n!)^2 \geq 2^{-n}(2n)!$ and $\|Y^n f\|^2 \leq \|f\| \, \|Y^{2n}f\|$, we see

$$\left(\|Y^n f\|\frac{t^n}{n!}\right)^2 \leq 2^{-n}\|f\| \, \|Y^{2n}f\|\frac{(2t)^{2n}}{(2n)!}$$

$$\leq 2^{-n}\|f\| \sum_{K=0}^{n}\binom{n}{K}\|\partial U(Y_1)^{2K}\partial U(Y_2)^{2(n-K)}f\|\frac{(2t)^{2n}}{(2n)!}.$$

Since $2^{-n}\sum_{K=0}^{n}\binom{n}{K} = 1$, this is bounded by (I.3).

Also, by (I.1), $\{\|(\partial/\partial t)^n f\| t^n/n!\}$ is bounded for all $t > 0$. Hence, by (I.5) and (I.1), f is entire for the representation of G^A considered in Example (I.8). Hence, by Goodman's result, $f = 0$.

It follows from this, as in Goodman [2], that no locally injective representation of G^α, $\mathrm{Re}\,\alpha \neq 0$, can have nonzero entire vectors.

If $\mathrm{Re}\,\alpha = 0$, every unitary representation of G^α has a dense set of entire vectors. This will follow from the general theory of §II.

II. In this section U will be a representation of G in a Hilbert space $\mathcal{H}$. Recall that G is said to be class R iff the eigenvalues of $\mathrm{ad}\,X$ for $X \in \mathcal{L}$ are all pure imaginary.

Proposition (II.1). *If U has a holomorphic extension, then G modulo the kernel of U is class R.*

Proof. If K is the connected component of the identity of the kernel of U, U induces a representation of G/K and clearly this representation has a holomorphic extension iff U does. Therefore, we may assume that U is locally injective. Then, by Examples (I.8) and (I.9), G can contain no analytic subgroups isomorphic to G^α ($\mathrm{Re}\,\alpha \neq 0$) or G^A. This implies that $\mathcal{L}$ is class R for if not, there is an $X \in \mathcal{L}$ and a $Y \in \mathcal{L}_c$ for which $[X,Y] = \alpha Y$ where $\alpha \in C$,

Re $\alpha \neq 0$. If $Y = Y_1 + iY_2$ and $[Y_1, Y_2] = 0$, then $\{X, Y_1, Y_2\}$ span a Lie algebra isomorphic to that of G^α, which is a contradiction. Hence $[Y_1, Y_2] \neq 0$. But then $[X, [Y_1, Y_2]] = [[X, Y_1], Y_2] + [Y_1, [X, Y_2]] = 2 \operatorname{Re} \alpha[Y_1, Y_2]$. Hence $\{X, [Y_1, Y_2]\}$ span a Lie algebra isomorphic to G^A which is a contradiction.   Q.E.D.

The converse of (II.1) is true, but the proof is more involved. We begin by considering the L^p left regular representation of G. We will call entire vectors for this representation L^p entire vectors. We will denote the L^2 left regular representation by L^G.

Recall that a real Lie algebra $\mathcal{L}$ is said to be semisimply split by a semidirect product decomposition $\mathcal{L} = \mathcal{T} + \mathcal{N}$ ($\mathcal{N}$ an ideal) provided $\mathcal{N}$ is nilpotent, $\mathcal{T}$ is abelian and ad $\mathcal{T} \mid \mathcal{N}$ is a semisimple family of linear transforms (i.e. there is a basis of $\mathcal{N}_c$ consisting of eigenvectors of the operators ad $\mathcal{T} \mid \mathcal{N}$. A semidirect product decomposition of G, $G = T \cdot N$, semisimply splits G provided the corresponding decomposition of $\mathcal{L}$ semisimply splits $\mathcal{L}$.

Lemma 1. *If G is class R and $G = T \cdot N$ is a semisimple splitting of G, then $\mathcal{H}_\infty^\omega(L^G)$ is dense in $L^2(G)$. In fact, if $f \in \mathcal{H}_\infty^\omega(L^T)$ and $g \in \mathcal{H}_\infty^\omega(L^N)$, then $f \otimes g \in \mathcal{H}_\infty^\omega(L^G)$ where $f \otimes g$ is the function $f \otimes g(tn) = f(t)g(n)$ for $t \in T$, $n \in N$.*

Proof. From the solvability of G, $G_c = T_c \cdot N_c$. Let f and g be as above. Since for $s_1, t \in T$ and $s_2, n \in N$,

$$(1) \qquad L^G(s_1 s_2)f \otimes g(tn) = (L^T(s_1)f) \otimes (L^N(t^{-1}s_1 s_2 s_1^{-1}t)g)(tn)$$

we may formally define an analytic extension of L^G at $f \otimes g$ via $L^G(z_1 z_2)f \otimes g(tn) = (L^T(z_1)f) \otimes L^N(t^{-1}z_1 z_2 z_1^{-1}t)g(tn)$ for $z_1 \in T_c$, $z_2 \in N_c$. We need to show that this remains in L^2. This follows from the fact that the adjoint action of T on N_c has relatively compact orbits, and hence $\|L^N(t^{-1}z_1 z_2 z_1^{-1}t)g\|_2$ is a uniformly bounded function of t. The relative compactness follows from the class R assumption for if $X_1, \ldots, X_n$ is a basis of N_c consisting of eigenvectors for ad $\mathcal{T} \mid \mathcal{N}_c$, then, for $Y \in T$, $c_i \in C$,

$$\exp Y \cdot \exp(c_1 X_1 + \cdots + c_n X_n) \cdot \exp -Y = \exp(c_1 e^{i\alpha_1} X_1 + \cdots + c_n e^{i\alpha_n} X_n),$$

where ad $Y(X_i) = i\alpha_i X_i$, α_i real. The required analyticity follows by an application of Morera's theorem and the verifiable fact that orbits of compact sets of N_c under the adjoint action of T on N_c are relatively compact.

Hence, to conclude our proof, we need only show that T and N have dense sets of L^2 entire vectors. However, since T and N are nilpotent, it follows from results of Goodman [3, Corollary (5.5)] that the L^2 left regular representation of T and N have dense sets of entire vectors.   Q.E.D.

We are now in a position to prove the converse of (II.1).

Theorem (II.2) *U has a holomorphic extension iff G modulo the kernel of U is class R.*

Proof. If f is any L^1 entire vector for L^G, then it follows by standard arguments that $w = \int_G f(g) U_g v \, dg$ is entire for U, the analytic extension being given by $U_z w = \int L^G_{z^{-1}} f(g) U_g(v) \, dg$. If G has a dense set of such f, then the linear span of the set of such w is dense, as can be seen by the usual annihilator argument. Hence, it suffices to show that G has a dense set of L^1 entire vectors. Since the pointwise product of two L^2 entire vectors is an L^1 entire vector (there is an obvious analytic extension) it suffices to consider L^G.

To do so, we use the semisimple splitting of Auslander-Moore [1, p. 132]. In particular, in [1] it was shown that G can be imbedded as a normal subgroup in a connected, simply connected solvable, Lie group G_0 which can be semisimply split by factors T and N (N normal). Furthermore $G_0 = T \cdot G$ and this decomposition is semidirect. G_0 can be chosen to be class R for if $\mathcal{L}_0$ is the Lie algebra of G_0, the collection $\mathfrak{I}$ of elements in $\mathcal{L}_0$ with pure imaginary eigenvalues is a subspace which contains $[\mathcal{L}_0, \mathcal{L}_0]$ and hence is a Lie algebra. It is easily shown that if $\mathcal{J}$ and $\mathcal{N}$ are the Lie algebras of T and N, then $\mathfrak{I} = \mathfrak{I} \cap \mathcal{J} + \mathcal{N}$ is a semisimple splitting of $\mathfrak{I}$. Also $\mathfrak{I} = \mathcal{J} \cap \mathfrak{I} + \mathcal{L}$. Hence, taking G_0 to be class R, we know, by Lemma 1, L^{G_0} has an L^2 dense set of entire vectors.

Now, since $G_0 = T \cdot G$, Haar measure on G_0 decomposes into a product of Haar measure on T and G, and $L^2(G_0)$ can be decomposed as a direct integral of $L^2(G)$ over T with respect to Haar measure. If $f \in L^2(G_0)$ and $t \in T$, the tth component of f is the map of G into C, $f_t(g) = f(tg)$. This decomposition decomposes $L^{G_0}|_G$ as $\int \oplus L^t dT(t)$ where $L^t(g) = L^G(t^{-1}gt)$. By Goodman's direct integral theorem [3, Lemma 3.1] it follows that almost every L^t must have dense sets of entire vectors. It is easily shown that any entire vector for L^t is entire for L^G. In fact $\mathcal{H}^\omega_\infty(L^t) = \mathcal{H}^\omega_\infty(L^G)$ for all $t \in T$. Q.E.D.

Remark. The procedure for producing entire vectors described above can be used to produce entire vectors having certain regularity properties. For example, let $\| \ \|$ be a left invariant Riemannian metric on G_0. Suppose f and g are as in Lemma 1. Suppose also that for all compact sets Ω in T and χ in N, there are positive constants B_Ω, B', C_χ, C', (B' and C' independent of Ω and χ) such that

$$(2) \qquad |L^T_{z_1} f(t)| \le B_\Omega \exp(-B'\|t\|^2) \quad \text{and} \quad |L^N_{z_2} g(n)| \le C_\chi \exp(-C'\|n\|^2)$$

for $z_1 \in \Omega$, $z_2 \in \chi$, $n \in N$, $t \in T$.

Then, from the above proof, it is easily seen that if Λ is a compact subset of G_0, then there are positive constants A_Λ and A' (A' independent of Λ) such that

$$|L_{z_1 z_2} f \otimes g(tn)| \le A_\Lambda \exp(-A'\|tn\|^2).$$

In fact, if Ω and χ are the projections of Λ in T_c and N_c respectively, and if χ_0 is the closure of the orbit of $\Omega \chi \Omega^{-1}$ by T in N_c (which is compact by the above proof) then by (1) and (2) and the fact that $\|tn\| \le \|t\| + \|n\|$

$$|L_{z_1 z_2} f \otimes g(tn)| \le B_\Omega(\exp(-B'\|t\|^2)) \cdot C_{\chi_0} \exp(-C'\|n\|^2)$$

$$\le B_\Omega C_{\chi_0} \exp(-A'\|tn\|^2)$$

where $A' = \frac{1}{4}\inf(B', C')$ and where $z_1 z_2 \in \Lambda$, $t \in T$, $n \in N$.

From the proof of (II.2), for a.e. $t \in T$, the map $x \to (f \otimes g)(tx) = (f \otimes g)$ (x) is, when restricted to G, entire for L'. Hence, from the decomposition of L^{G_0} as a direct integral of the L'_t, $t \in T$ (see (II.2)) we see that for $z \in G_c$,

$$L^G(z)((f \otimes g)_t) = L'(tzt^{-1})((f \otimes g)_t)$$

$$= [L^{G_0}(tzt^{-1})(f \otimes g)]_t = L^{G_0}(zt^{-1})(f \otimes g).$$

Thus, if Δ is a compact subset of G_c, then for $k \in G$, $z \in \Delta$,

$$|L^G(z)((f \otimes g)_t)(k)| = |L^{G_0}(zt^{-1})(f \otimes g)(k)|$$

$$\leq A_{\Delta t^{-1}}\exp(-A'\|k\|^2).$$

Thus $(f \otimes g)_t$ satisfies an estimate similar to (2) above.

Entire vectors satisfying (2) always exist for nilpotent groups. In fact, let N be any c.s.c. nilpotent group and let $\mathcal{N} = \mathcal{L}(N)$. Let $\{X_j\}$, $j = 1, \ldots, d$, be an ordered Jordan Hölder basis for $\mathcal{N}$ considered as elements of $\mathcal{N}_c$. For $z \in C$, set $g_j(z) = \exp zX_j \in N_c$. For $P \in N^d$, $P = (P_1, \ldots, P_d)$, let $\mathcal{A}_P$ be the space of entire functions f on G_c for which the function $F(z_1, \ldots, z_d)$ $= f(\exp z_1 X_1 \cdots \exp z_d X_d)$ satisfies the hypothesis of Theorem (5.1) of [3], i.e. there are positive constants A, B and C for which

$$|F(z_1, \cdots, z_d)| < C\exp\left\{-A\left(\sum |\text{Re } z_j|^{P_j}\right) + B\left(\sum |\text{Im } z_j|^{P_j}\right)\right\}.$$

Goodman showed that there is a positive constant m such that if $P_{K+1} > P_K m$ for all K, then $f \in \mathcal{A}_P$ implies that there are positive constants C_1, A_1, D, q such that

$$f(g_j(w)g_1(t_1)\cdots g_d(t_d)) \leq C_1\exp\left\{-A_1\sum_{i=1}^{d}|t_i|^{P_i} + D|w|^q\right\}$$

for all $t_i \in R$, $w \in C$.

An obvious modification of Goodman's proof shows that there are constants A_1, B_1, C_1, D, q (independent of w and z_i) such that

$$|f(g_j(w)g_1(z_1)\cdots g_d(z_d))| \leq C_1\exp\left\{-A_1\sum|\text{Re } z_j|^{P_j} + B_1\sum|\text{Im } z_j|^{P_j} + D|w|^q\right\}$$

for $z_i, w \in C$.

It follows that $\mathcal{A}_P$ is invariant under complex left translations and that for $w = g_1(w_1)\cdots g_d(w_d)$, $w_i \in C$,

$$|f(wg_1(z_1)\cdots g_d(z_d))| \leq C\exp\left\{-A\sum|\text{Re } z_i|^{P_i} + B\sum|\text{Im } z_i|^{P_i} + D\sum|w_i|^q\right\}.$$

If $P_1 \geq 2$, it follows that the restriction of f to G satisfies (2) above. (Note that $\|g_1(t) \cdots g_d(t_d)\| \leq \|g_1(t_1)\| + \cdots + \|g_d(t_d)\| \leq |t_1| + \cdots + |t_d|$.)   Goodman showed that $\mathcal{A}_P$ is a dense subset of $L^2(G)$.

Corollary (II.3). *Suppose U is as in the introduction and G is class R. Then if U has a dense set of vectors v for which $g \rightarrow U_g v e^{M\|g\|}$ is bounded for some $M < 0$ (depending on v), then U has a dense set of entire vectors.*

Proof. This follows as in Moore [7, Theorem 5, Chapter 6], where instead of the family of functions $f(t, \cdot)$, we regularize with respect to entire vectors satisfying (2) above.

Corollary. *Every representation of G in a Banach space has a dense set of entire vectors.*

Proof. Banach space representations are known to satisfy the hypothesis of the previous corollary with respect to every vector.

Remarks. The proof Moore's theorem cited above is somewhat more complicated than required in our case as we can prove existence of the required integral via a Riemannian sum since our representation is given as being continuous.

Our results on unitary representations admit a nice description in terms of the Green kernel. (After Leon Green who, we are told, has studied this kernel extensively.)

Definition. Let $\mathcal{L}$ be a Lie algebra. The Green kernel of $\mathcal{L}$ is the smallest ideal $\mathcal{K}$ such that $\mathcal{L}/\mathcal{K}$ is type R. If G is the connected, simply connected Lie group corresponding to $\mathcal{L}$, then the Green kernel of G is the analytic subgroup of G corresponding to $\mathcal{K}$.

Then (2.2) may be paraphrased as

Corollary (II.4). *A unitary representation U of a solvable Lie group has a dense set of entire vectors iff U is the identity on the Green kernel. In any event, the closure of the set of entire vectors is the space of vectors left invariant by the Green kernel.*

This corollary solves completely the question of when unitary representations of solvable Lie groups have entire vectors. However, as noted in Remark (I.7), the concept of entire vector is definable for nonsolvable groups.

To consider the general problem, let G be a not necessarily solvable connected, simply connected Lie group. By the Levi decomposition G is a semidirect product of a semisimple group T and a normal solvable group S. If U is any locally injective unitary representation of G which has a dense set of entire vectors, the restriction of U to S has a dense set of entire vectors. Hence S is type R. Similarly, by a theorem of Goodman (Goodman [2, Theorem 8.1]), T must be compact.

Conversely, suppose G is a semidirect product of a compact semisimple group T and a normal solvable type R group S. Since finite dimensional representations have bounded differentials, every vector for such a representation is entire.

Hence, by the Peter-Weyl theorem there is a dense set of L^2 entire vectors for T. If f and g are L^2 entire vectors for T and S respectively, it follows as in Lemma 1 above that $f \otimes g$ is entire for $L^G \mid S$. It follows from formula (1) that $f \otimes g$ is entire for $L^G \mid T$ also. Hence from Theorem (I.4) (note that the proof of this theorem was "infinitesimal" and hence did not depend on the solvability of the group), $f \otimes g$ is an L^2 entire vector for G. Thus G has a dense set of L^2 entire vectors. It is not difficult to show from the Leibniz product theorem for derivatives that the pointwise product of L^2 entire vectors is an L^1 entire vector. Finally, if U is a unitary representation of G and f is an L^1 entire vector for G, then $w = \int_G f(g) U_g v \, dg$ is an entire vector for all v in the representation space as follows from the identity

$$\partial U(X) w = \int_{G_0} (-\partial L(X) f)(g) U_g v \, dg$$

for all X in the Lie algebra of G. We have then shown that every unitary representation of G has a dense set of entire vectors. Such G can be characterized simply as the not necessarily solvable class R Lie groups due to the following lemma.

Lemma. *A Lie algebra $\mathcal{L}$ is class R iff $\mathcal{L}$ is a semidirect product of a compact semisimple Lie algebra $\mathcal{T}$ and a type R solvable Lie algebra $\mathcal{S}$.*

Proof. Let G be the c.s.c. Lie group corresponding to $\mathcal{L}$. If $\mathcal{L}$ is such a semidirect product, from the above we see that every unitary representation of G must have dense sets of entire vectors. (II.1) then shows that G must be type R. (Note that the proof of (II.1) nowhere assumed solvability of G.)

Conversely, if $\mathcal{L}$ is type R, let $\mathcal{L} = \mathcal{T} + \mathcal{S}$ be the Levi decomposition ($\mathcal{T}$ semisimple and $\mathcal{S}$ solvable). Since $\mathcal{S}$ is automatically type R, it suffices to show that $\mathcal{T}$ is compact. Let B be the Killing form for $\mathcal{T}$ and let $X \in \mathcal{T}$. Since $\operatorname{tr}(\operatorname{ad} X \operatorname{ad} X) = B(X, X)$ is the sum of the squares of the eigenvalues of $\operatorname{ad} X$ weighted according to multiplicity, $B(X, X) \leq 0$. Thus, since B is nondegenerate, B is negative definite and $\mathcal{T}$ is compact.   Q.E.D.

Corollary (II.5). *Corollary (II.4) is true with "solvable" omitted.*

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